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The generalized symmetry method for discrete equations

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Abstract
The generalized symmetry method is applied to a class of completely discrete equations including the Adler–Bobenko–Suris list. Assuming the existence of a generalized symmetry, we derive a few integrability conditions suitable for testing and classifying the equations of this class. Those conditions are used at the end to test for integrability discretizations of some well-known hyperbolic equations.

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1. Introduction

The discovery of new two-dimensional integrable partial difference equations (or $\mathbb{Z}^2$-lattice equations) is always a very challenging problem as, by proper continuous limits, many other results on integrable differential–difference and partial differential equations may be obtained.

The basic theory and results in nonlinear integrable differential equations can be found, for example, in the *Encyclopedia of Mathematical Physics* [18] or in the *Encyclopedia of Nonlinear Science* [19].

The classification of integrable nonlinear partial differential equations has been widely discussed in many relevant papers. Let us just mention here the classification scheme introduced by Shabat, where the formal symmetry approach has been introduced (see [37, 38] for a review). This approach has been successfully extended to the differential–difference case by Yamilov [8, 35, 57, 58]. In the completely discrete case, the situation turns out to be quite different, and, up to now, the formal symmetry technique has not been able to provide any result.

In the case of difference–difference equations, many important results on integrable models are contained in [3, 16, 24, 33, 43, 45]. The first classification results have been obtained by Adler [5] and by Adler *et al* for linear–affine equations [6, 7]. The so obtained equations have been thoroughly studied by many researchers, and it has been shown that they have Lax pairs and possess generalized symmetries [32, 40, 47, 52, 53]. Moreover, these partial difference equations can be seen as Bäcklund transformations of the well-known integrable nonlinear differential–difference equations of the Volterra type [11, 28, 29, 32, 42, 43].

We study, in this paper, the following class of autonomous discrete equations on the lattice $\mathbb{Z}^2$:

$$u_{i+1,j+1} = F(u_{i+1,j}, u_{i,j}, u_{i,j+1}), \tag{1}$$

where i and j are the arbitrary integers. Many integrable examples of the equations of this form are known [6, 30, 31, 53, 54]. There are a number of papers which discuss various schemes for testing [2, 12, 15, 20, 22, 23, 46] and classifying [6, 9, 21, 49] integrable equations of the form (1). In [9, 21] the classification of linearizable equations is considered; in [2, 12, 15, 20, 22, 23, 46] extensions of the Painlevé test are carried over to the discrete case, while in [6, 49] equations which have a Lax pair and thus can be integrated by the inverse scattering method are discussed. Requiring additional geometrical symmetry properties, a classification result has been obtained in [6] together with a list of integrable equations. However, the symmetries for those discrete equations obtained in [47, 53] show that the obtained class of equations contained in [6] is somehow restricted [32]. From [32] it follows that one should expect a larger number of integrable discrete equations of the kind of equation (1) than those up to now known.

Equations (1) are possible discrete analogs of the hyperbolic equations

$$u_{x,y} = F(u_x, u, u_y). \tag{2}$$

Equations (2) are very important in many fields of physics, and, as such, they have been studied using the generalized symmetry method, however without much success. Only the following two particular cases:

$$u_{x,y} = F(u), \tag{3}$$

$$u_x = F(u, v), \qquad v_y = G(u, v), \tag{4}$$

which are essentially easier, have been solved [60, 61]. The study of the class of equations (1) may be important to characterize the integrable subcases of equation (2).

In section 2, we introduce and discuss some necessary notions, such as generalized symmetries and conservation laws for discrete systems of the form (1), and in section 3, as a motivation for the use of this approach, we show that one can construct a partial difference equation closely related to the modified Volterra equation, which does not belong to the ABS class of equations as it is not 3D consistent around the cube in the strict sense of [6] and does not have the D_4 symmetry. In section 4, following the standard scheme of the generalized symmetry method, we derive a few integrability conditions for the class (1). These conditions are not sufficient to carry out a classification of the discrete equations (1). So in section 5, we consider just five-point generalized symmetries. This provides further integrability conditions. With these extra conditions, the set of obtained conditions will be suitable for testing and classifying simple classes of the difference equations of the form (1). As an example, in section 6, we apply these conditions to the class of equations

$$u_{i+1,j+1} = u_{i+1,j} + u_{i,j+1} + \varphi(u_{i,j}), \tag{5}$$

a trivial approximation to the class (3). This calculation is an example of the classification problem for a class depending on one unknown function of one variable. This class contains trivial approximations of some well-known integrable equations included in the class (3), namely, the sine-Gordon, Tzitzèika and Liouville equations. Section 7 contains some conclusive remarks.

2. Preliminary definitions

As equation (1) has no explicit dependence on the point (i, j) of the lattice, we thus assume that the same will be true for the generalized symmetries and conservation laws we will be considering in the following. For this reason, without loss of generality, we write down symmetries and conservation laws at the point $(0, 0)$. Thus equation (1) can be written as

$$u_{1,1} = f_{0,0} = F(u_{1,0}, u_{0,0}, u_{0,1}). \tag{6}$$

Whenever convenient we will express our formulas in terms of the two shift operators, T_1 and T_2:

$$T_1 u_{i,j} = u_{i+1,j}, \qquad T_2 u_{i,j} = u_{i,j+1}. \tag{7}$$

To get a scheme which is invertible and to provide propagation in both discrete directions, we have to suppose that the function F depends on all its variables, i.e.

$$\partial_{u_{1,0}} F \cdot \partial_{u_{0,0}} F \cdot \partial_{u_{0,1}} F \neq 0. \tag{8}$$

The functions $u_{i,j}$ are related among themselves by equation (6) and its shifted values

$$u_{i+1,j+1} = T_1^i T_2^j f_{0,0} = f_{i,j} = F(u_{i+1,j}, u_{i,j}, u_{i,j+1}),$$

and it is easy to see that all of them can be expressed in terms of the functions

$$u_{i,0}, \quad u_{0,j}, \tag{9}$$

where i, j are arbitrary integers. This is not the only possible choice of independent variables [10], but, being the simplest, is the one we will use in the following. The functions (9) play the role of boundary initial conditions for equation (6).

The evolutionary form of a generalized symmetry of equation (6) is given by the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{0,0} = g_{0,0} = G(u_{n,0}, u_{n-1,0}, \ldots, u_{n',0}, u_{0,k}, u_{0,k-1}, \ldots, u_{0,k'}), \tag{10}$$

where $n \geqslant n'$, $k \geqslant k'$. This form of this equation at the various points of the lattice is obtained by applying the shift operators T_1 and T_2:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{i,j} = T_1^i T_2^j g_{0,0} = g_{i,j} = G(u_{i+n,j}, \ldots, u_{i+n',j}, u_{i,j+k}, \ldots, u_{i,j+k'}).$$

Equation (10) is a generalized symmetry of equation (6) if the two equations (6) and (10) are compatible for all independent variables (9), i.e.

$$\frac{\mathrm{d}u_{1,1}}{\mathrm{d}t} - \left.\frac{\mathrm{d}f_{0,0}}{\mathrm{d}t}\right|_{u_{1,1}=f_{0,0}} = 0. \tag{11}$$

In practice, equation (11) reads

$$g_{1,1} = \left(g_{1,0}\partial_{u_{1,0}} + g_{0,0}\partial_{u_{0,0}} + g_{0,1}\partial_{u_{0,1}}\right) f_{0,0}. \tag{12}$$

The condition that generalized symmetries do not depend explicitly on the lattice point (i, j) is natural, as integrable autonomous equations possess infinite hierarchies of autonomous symmetries. On the other hand, if we introduce a (i, j) dependence in the generalized symmetries and in the class of equations (1), the problem becomes more complicated, as the study of differential–difference equations has shown (cf [58] for the autonomous case and [35, 59] for the non-autonomous one).

Equation (12) must be identically satisfied when all the variables $u_{i,j}$ contained in the functions $g_{i,j}$ and in the derivatives of $f_{0,0}$ are expressed in terms of the independent variables (9). This result provides strict conditions, given by a set of equations for the functions F and G, often overdetermined.

Let us consider some autonomous functions $p_{0,0}, q_{0,0}$ which depend on a finite number of functions $u_{i,j}$ and have no explicit dependence on the point (i, j) of the lattice. The relation

$$(T_1 - 1)p_{0,0} = (T_2 - 1)q_{0,0} \tag{13}$$

is called a (local i, j-independent) conservation law of equation (6) if it is satisfied on the solutions of this equation. To check it, we need to express all variables in terms of the independent variables (9) and require that it is identically satisfied.

Starting from the choice of the independent variables (9) and the class of autonomous difference equations (6), we can prove a few useful statements which will be used for studying the compatibility condition (12). Let us consider the functions $u_{i,1}, u_{1,j}$ appearing in equation (12). We can prove the following theorem.

Theorem 1. *The transformation $\mathcal{T} : (u_{i,0}, u_{0,j}) \to (\tilde{u}_{i,0}, \tilde{u}_{0,j})$, given by the shift operator T_2*

$$\tilde{u}_{0,j} = u_{0,j+1}, \qquad \tilde{u}_{i,0} = u_{i,1}, \quad i \neq 0, \tag{14}$$

is invertible whenever equation (6) is satisfied. Moreover, if a function ϕ is nonzero, then $T_2\phi \neq 0$ too.

Proof. The invertibility of the transformation $\tilde{u}_{0,j} = u_{0,j+1}$ is obvious. Let us show by induction that for any $i \geqslant 1$:

$$\tilde{u}_{i,0} = \tilde{u}_{i,0}(u_{i,0}, u_{i-1,0}, \ldots, u_{1,0}, u_{0,0}, u_{0,1}), \qquad \partial_{u_{i,0}}\tilde{u}_{i,0} \neq 0, \quad \partial_{u_{0,1}}\tilde{u}_{i,0} \neq 0. \tag{15}$$

It follows from equation (6) and condition (8) that the proposition is true for $\tilde{u}_{1,0} = u_{1,1}$. For $i \geqslant 1$, from equation (6), we get

$$\tilde{u}_{i+1,0} = u_{i+1,1} = F(u_{i+1,0}, u_{i,0}, \tilde{u}_{i,0}), \tag{16}$$

with $\tilde{u}_{i,0}$ given by equation (15). So $\tilde{u}_{i+1,0}$ has the same structure as $\tilde{u}_{i,0}$, and thus equation (15) is true. As $\tilde{u}_{i,0}$ depends on $u_{0,1}$, then the functions $u_{i+1,0}, u_{i,0}, \tilde{u}_{i,0}$ are functionally independent, i.e. $\partial_{u_{i+1,0}}\tilde{u}_{i+1,0} \neq 0$ and $\partial_{u_{0,1}}\tilde{u}_{i+1,0} \neq 0$. A similar analysis can be carried out in the case of the functions $\tilde{u}_{i,0}$ when $i \leqslant -1$. In this case, we have

$$\tilde{u}_{i,0} = \tilde{u}_{i,0}(u_{i,0}, u_{i+1,0}, \ldots, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad \partial_{u_{i,0}}\tilde{u}_{i,0} \neq 0, \quad \partial_{u_{0,1}}\tilde{u}_{i,0} \neq 0. \tag{17}$$

From equations (15) and (17) it follows that the transformation (14) is invertible.

To prove the second part of this theorem , let us consider a non-constant function $\phi \neq 0$. Taking into account equation (6) and its shifted values, ϕ can always be expressed in terms of the independent variables as

$$\phi = \Phi(u_{N,0}, u_{N-1,0}, \ldots, u_{N',0}, u_{0,K}, u_{0,K-1}, \ldots, u_{0,K'}), \tag{18}$$

for some integer numbers N, N', K and K' such that $N \geqslant N'$, $K \geqslant K'$. Then we will have

$$T_2\phi = \Phi(\tilde{u}_{N,0}, \ldots, \tilde{u}_{N',0}, \tilde{u}_{0,K}, \ldots, \tilde{u}_{0,K'}). \tag{19}$$

If ϕ depends essentially on the variables $u_{i,0}$ with $i \neq 0$, then there must exist two numbers N and N' such that $\partial_{u_{N,0}}\phi \neq 0$ and $\partial_{u_{N',0}}\phi \neq 0$. When $N > 0$, from equation (15) it follows that only the function $\tilde{u}_{N,0}$ appearing in equation (19) depends on $u_{N,0}$. Hence $\partial_{u_{N,0}} T_2\phi \neq 0$, i.e. $T_2\phi \neq 0$. The case, when $N' < 0$, is analogous. If ϕ depends only on $u_{0,j}$, then $\partial_{u_{0,K}}\phi \neq 0$ and $\partial_{u_{0,K'}}\phi \neq 0$, and the proof is obvious. □

The operators T_1, T_1^{-1}, T_2^{-1} act on the variables (9) in an analogous way. Consequently, they also define invertible transformations. As a result, we can state the following proposition.

Proposition 1. *For any nonzero function ϕ, $T_1^l T_2^m \phi \neq 0$ for any $l, m \in \mathbb{Z}$.*

From equations (14), (15) and (17) we can derive the structure of some of the partial derivatives of the functions $u_{i,1}$. For convenience, from now on, we will define

$$f_{u_{i,j}} = \partial_{u_{i,j}} f_{0,0}, \qquad g_{u_{i,j}} = \partial_{u_{i,j}} g_{0,0} \tag{20}$$

for the derivatives of the functions $f_{0,0}$ and $g_{0,0}$ appearing in equations (6) and (10). Then, for example, from equation (6) we get $\partial_{u_{1,0}} u_{1,1} = f_{u_{1,0}}$. For $i > 0$, from equations (15) and (16) it follows that $\partial_{u_{i+1,0}} u_{i+1,1} = T_1^i \partial_{u_{1,0}} u_{1,1}$. From equation (6), we can also get $u_{-1,1} = \hat{F}(u_{-1,0}, u_{0,0}, u_{0,1})$ and then by differentiation

$$\partial_{u_{-1,0}} u_{-1,1} = -T_1^{-1} \frac{f_{u_{0,0}}}{f_{u_{0,1}}}. \tag{21}$$

Then, applying the operator T_1^{i+1}, with $i < 0$, to equation (21) it follows that $\partial_{u_{i,0}} u_{i,1} = -T_1^i \frac{f_{u_{0,0}}}{f_{u_{0,1}}}$. For the functions of the form $u_{1,j}$ we get similar results. So we can state the following proposition.

Proposition 2. *The functions $u_{i,1}$, $u_{1,j}$ are such that*

$$\begin{aligned}
&i > 0: \quad u_{i,1} = u_{i,1}(u_{i,0}, u_{i-1,0}, \ldots, u_{1,0}, u_{0,0}, u_{0,1}), \quad \partial_{u_{i,0}} u_{i,1} = T_1^{i-1} f_{u_{1,0}};\\
&i < 0: \quad u_{i,1} = u_{i,1}(u_{i,0}, u_{i+1,0}, \ldots, u_{-1,0}, u_{0,0}, u_{0,1}), \quad \partial_{u_{i,0}} u_{i,1} = -T_1^{i} \frac{f_{u_{0,0}}}{f_{u_{0,1}}};\\
&j > 0: \quad u_{1,j} = u_{1,j}(u_{1,0}, u_{0,0}, u_{0,1}, \ldots, u_{0,j-1}, u_{0,j}), \quad \partial_{u_{0,j}} u_{1,j} = T_2^{j-1} f_{u_{0,1}};\\
&j < 0: \quad u_{1,j} = u_{1,j}(u_{1,0}, u_{0,0}, u_{0,-1}, \ldots, u_{0,j+1}, u_{0,j}), \quad \partial_{u_{0,j}} u_{1,j} = -T_2^{j} \frac{f_{u_{0,0}}}{f_{u_{1,0}}}.
\end{aligned} \tag{22}$$

3. Integrable example

In this section we show, using a simple example, that effectively there are integrable equations which possess hierarchies of the generalized symmetries of the form postulated in equation (10) and are not included in the ABS lists.

As is well known [55], the modified Volterra equation

$$u_{i,t} = \left(u_i^2 - 1\right)(u_{i+1} - u_{i-1}) \tag{23}$$

is transformed into the Volterra equation $v_{i,t} = v_i(v_{i+1} - v_{i-1})$ by two discrete Miura transformations:

$$v_i^{\pm} = (u_{i+1} \pm 1)(u_i \mp 1). \tag{24}$$

For any solution u_i of equation (23), one obtains by the transformations (24) two solutions v_i^+ and v_i^- of the Volterra equation. From a solution of the Volterra equation v_i, one obtains two solutions u_i and $\tilde{u}_i$ of the modified Volterra equation. The composition of the Miura transformations (24)

$$v_i = (u_{i+1} + 1)(u_i - 1) = (\tilde{u}_{i+1} - 1)(\tilde{u}_i + 1) \tag{25}$$

provides a Bäcklund transformation for equation (23). Equation (25) allows one to construct, starting with a solution u_i of the modified Volterra equation (23), a new solution $\tilde{u}_i$.

Introducing for any index i, $u_i = u_{i,j}$ and $\tilde{u}_i = u_{i,j+1}$, where j is a new index, we can rewrite the Bäcklund transformation (25) as an equation of the form (1). At the point $(0, 0)$, it reads

$$(u_{1,0} + 1)(u_{0,0} - 1) = (u_{1,1} - 1)(u_{0,1} + 1). \tag{26}$$

Equation (26) does not belong to the ABS classification, as it is not invariant under the exchange of i and j and does not satisfy the 3D consistency property in the sense of [6]. It may be 3D consistent around the cube in accordance with the extended definition of [7] (cf [44]), but we leave this problem for a future work. To our knowledge, equation (26) has been introduced in [41] in a slightly different form together with a Lax pair, and nothing more has been known. A different Lax pair for this equation has been constructed in the recent paper [36]. One more Lax pair for a more general form of equation (26) is presented below. This equation is just an illustrative example for the present paper. That is why we are mainly interested here in its symmetry structure. Study of different properties, as e.g. relationships to the other known integrable equations, is also left for a future work.

The modified Volterra equation (23) can be interpreted as a three-point generalized symmetry of equation (26) involving only shifts in the i direction:

$$u_{0,0,t} = \left(u_{0,0}^2 - 1\right)(u_{1,0} - u_{-1,0}). \tag{27}$$

There also exists a generalized symmetry involving only shifts in the j direction, given by

$$u_{0,0,\tau} = \left(u_{0,0}^2 - 1\right)\left(\frac{1}{u_{0,1} + u_{0,0}} - \frac{1}{u_{0,0} + u_{0,-1}}\right), \tag{28}$$

which belongs, together with equation (27), to the complete list of the integrable Volterra-type equations presented in [57, 58]. Both equations have a hierarchy of generalized symmetries which, by construction, must be compatible with equation (26). Symmetries of equation (27) can be obtained in many ways, see e.g. [58]. Symmetries of equation (28) can be constructed using the master symmetry presented in [14]. The simplest generalized symmetries of equations (27) and (28) are given by the following equations:

$$u_{0,0,t'} = \left(u_{0,0}^2 - 1\right)\left(\left(u_{1,0}^2 - 1\right)(u_{2,0} + u_{0,0}) - \left(u_{-1,0}^2 - 1\right)(u_{0,0} + u_{-2,0})\right),$$

$$u_{0,0,\tau'} = \frac{u_{0,0}^2 - 1}{(u_{0,1} + u_{0,0})^2}\left(\frac{u_{0,1}^2 - 1}{u_{0,2} + u_{0,1}} + \frac{u_{0,0}^2 - 1}{u_{0,0} + u_{0,-1}}\right) - \frac{u_{0,0}^2 - 1}{(u_{0,0} + u_{0,-1})^2}\left(\frac{u_{0,0}^2 - 1}{u_{0,1} + u_{0,0}} + \frac{u_{0,-1}^2 - 1}{u_{0,-1} + u_{0,-2}}\right).$$

As can be checked by direct calculation, these equations are five-point symmetries of equation (26).

Moreover, equation (26) possesses two conservation laws (13) characterized by the following functions $p_{0,0}, q_{0,0}$:

$$p_{0,0}^{+} = \log\frac{u_{0,0} + u_{0,1}}{u_{0,0} + 1}, \qquad q_{0,0}^{+} = -\log(u_{0,0} + 1), \tag{29}$$

$$p_{0,0}^{-} = \log\frac{u_{0,0} + u_{0,1}}{u_{0,1} - 1}, \qquad q_{0,0}^{-} = \log(u_{0,0} - 1). \tag{30}$$

It is easy to check that equation (13) is identically satisfied on the solutions of equation (26) when we introduce into it the functions (29) or (30). Equation (26) possess also non-autonomous conservation laws; however, conservation laws of this kind will not be discussed here.

A more general form of both equations (25) and (26) is given by

$$v_{i,j} = (u_{i+1,j} + \alpha_j)(u_{i,j} - \alpha_j) = (u_{i+1,j+1} - \alpha_{j+1})(u_{i,j+1} + \alpha_{j+1}), \tag{31}$$

where α_j is a j-dependent function. For any j, the function $u_{i,j}$ satisfies the modified Volterra equation

$$u_{i,j,t} = \left(u_{i,j}^2 - \alpha_j^2\right)(u_{i+1,j} - u_{i-1,j})$$

depending on the function α_j. Function $v_{i,j}$, for any j, is a solution of the Volterra equation. Using equation (31) and starting from an initial solution $v_{i,0}$, we can construct new solutions of the Volterra equation:

$$v_{i,0} \to u_{i,1} \to v_{i,1} \to u_{i,2} \to v_{i,2} \to \cdots.$$

The Lax pair for equation (31) is given by

$$L_{i,j} = \begin{pmatrix} \lambda - \lambda^{-1} & -v_{i,j} \\ 1 & 0 \end{pmatrix},$$

which corresponds to the standard scalar spectral problem of the Volterra equation written in the matrix form, and by

$$A_{i,j} = \frac{1}{u_{i,j+1} - \alpha_{j+1}} \begin{pmatrix} (\lambda - \lambda^{-1})(u_{i,j+1} - \alpha_{j+1}) & 2\alpha_{j+1}\left(u_{i,j+1}^2 - \alpha_{j+1}^2\right) \\ -2\alpha_{j+1} & (\lambda - \lambda^{-1})(u_{i,j+1} + \alpha_{j+1}) \end{pmatrix}.$$

This Lax pair satisfies the Lax equation $A_{i+1,j}L_{i,j} = L_{i,j+1}A_{i,j}$.

Equation (31) is a direct analog of the well-known dressing chain

$$u_{j+1,x} + u_{j,x} = u_{j+1}^2 - u_j^2 + \alpha_{j+1} - \alpha_j, \tag{32}$$

which provides a way of constructing potentials $v_j = u_{j,x} - u_j^2 - \alpha_j$ for the Schrödinger spectral problem [50, 51]. The Lax pair given above is analogous to that of equation (32) presented in [51].

4. Derivation of the integrability conditions

In this section, following the standard scheme of the generalized symmetry method, we derive from the compatibility condition (12) four conditions necessary for the integrability of equation (6).

For a generalized symmetry (10), we suppose that if $g_{0,0}$ depends on at least one variable of the form $u_{i,0}$, then $g_{u_{n,0}} \neq 0$ and $g_{u_{n',0}} \neq 0$, and the numbers n and n' are called the orders of the symmetry. The same can be said about the variables $u_{0,j}$ and the corresponding numbers k, k' if $g_{u_{0,k}} \neq 0$ and $g_{u_{0,k'}} \neq 0$.

Theorem 2. *Let equation (6) possess a generalized symmetry (10) of orders n, n', k and k'. Then the following relations must take place:*

$$\textit{If} \quad n > 0 \quad \Longrightarrow \quad \left(T_1^n - 1\right) \log f_{u_{1,0}} = (1 - T_2)T_1 \log g_{u_{n,0}}; \tag{33}$$

$$\textit{If} \quad n' < 0 \quad \Longrightarrow \quad \left(T_1^{n'} - 1\right) \log \frac{f_{u_{0,0}}}{f_{u_{0,1}}} = (1 - T_2) \log g_{u_{n',0}}; \tag{34}$$

$$\textit{If} \quad k > 0 \quad \Longrightarrow \quad \left(T_2^k - 1\right) \log f_{u_{0,1}} = (1 - T_1)T_2 \log g_{u_{0,k}}; \tag{35}$$

$$\textit{If} \quad k' < 0 \quad \Longrightarrow \quad \left(T_2^{k'} - 1\right) \log \frac{f_{u_{0,0}}}{f_{u_{1,0}}} = (1 - T_1) \log g_{u_{0,k'}}. \tag{36}$$

Before going over to the proof of this theorem, let us clarify its meaning by noting that in the case of a three-point symmetry with $g_{0,0} = G(u_{1,0}, u_{0,0}, u_{-1,0})$, for which $n > 0$ and $n' < 0$, one can use both relations (33) and (34).

Proof. Let us consider the compatibility condition (12) expressed in terms of the independent variables (9). As $g_{0,0}$ depends on $u_{i,0}$ and $u_{0,j}$, the functions $(g_{1,1}, g_{1,0}, g_{0,1})$ depend on $(u_{i,1}, u_{1,j})$, whose form is given by proposition 2. Moreover, equation (12) will contain $u_{i,0}$ with $n + 1 \geqslant i \geqslant n'$ and $u_{0,j}$ with $k + 1 \geqslant j \geqslant k'$.

If $n > 0$, applying to equation (12) the operator $\partial_{u_{n+1,0}}$ and using the results (22) contained in proposition 2, we get

$$T_1 T_2 (g_{u_{n,0}}) T_1^n f_{u_{1,0}} = f_{u_{1,0}} T_1 g_{u_{n,0}}.$$

Applying the logarithm to both sides of the previous equation, we obtain equation (33). The other cases are obtained in a similar way by differentiating equation (12) with respect to $u_{n',0}$, $u_{0,k+1}$ and $u_{0,k'}$. □

Equations (33)–(36) can be expressed as a standard conservation law of the form (13), using the obvious well-known formulas

$$T_l^m - 1 = (T_l - 1)\left(1 + T_l + \cdots + T_l^{m-1}\right), \qquad m > 0,$$
$$T_l^m - 1 = (1 - T_l)\left(T_l^{-1} + T_l^{-2} + \cdots + T_l^{m}\right), \qquad m < 0,$$

where $l = 1, 2$. This means that, from the existence of a generalized symmetry, one can construct some conservation laws.

Theorem 2 provides integrability conditions, i.e. that for an integrable equation there must exist a function $g_{0,0}$ satisfying equations (33)–(36). The unknown function $g_{0,0}$ must depend on a finite number of independent variables. These integrability conditions turn out to be difficult to use for testing and classifying difference equations.

In the case of the differential–difference equations of Volterra or Toda type [58], there are integrability conditions equivalent to equations (33)–(36). In order to check these integrability conditions, one can use the formal variational derivatives [17, 26, 56, 58], defined as

$$\frac{\delta^{(1)}\phi}{\delta u_{0,0}} = \sum_{i=-N}^{-N'} \frac{\partial T_1^i \phi}{\partial u_{0,0}}, \qquad \frac{\delta^{(2)}\phi}{\delta u_{0,0}} = \sum_{j=-K}^{-K'} \frac{\partial T_2^j \phi}{\partial u_{0,0}},$$

for ϕ given by equation (18). Using such variational derivatives, for example, the integrability conditions (33) and (35) are reduced to the following equations:

$$\frac{\delta^{(2)}}{\delta u_{0,0}}\left(T_1^n - 1\right) \log f_{u_{1,0}} = 0, \qquad \frac{\delta^{(1)}}{\delta u_{0,0}}\left(T_2^k - 1\right) \log f_{u_{0,1}} = 0, \tag{37}$$

which do not involve any unknown function. This result is due to the fact that in this case all discrete variables are independent. In a completely discrete case, the situation is essentially different. Some of the discrete variables are dependent and the variational derivatives must be calculated modulo equation (1). So equations (37) will no longer be valid. If we apply here the variational derivatives, we will get, at most, some partial results depending on the choice of the independent variables introduced.

The conservation laws (33)–(36) depend on the order of the symmetry. These conservation laws can be simplified under some assumptions on the structure of the Lie algebra of the generalized symmetries. If we assume that for a given equation we are able to get generalized symmetries for any value of n and k, then we can derive order-independent conservation laws, using a trick standard in the generalized symmetry method [58]. This assumption implies that if, for example, we have a generalized symmetry of order n then there must also be one of order $n + 1$. This is a very constraining assumption which is not always verified, as we know from the continuous case [38]. Here it is used just as an example for the construction of simplified formulas. In fact such simplified formulas can be obtained assuming any difference between the orders of two generalized symmetries, and in the following section we consider an example with difference 2.

So, in the following theorem, we will assume that in addition to (10) a second generalized symmetry

$$u_{0,0,\tilde{t}} = \tilde{g}_{0,0} = \tilde{G}(u_{\tilde{n},0}, u_{\tilde{n}-1,0}, \ldots, u_{\tilde{n}',0}, u_{0,\tilde{k}}, u_{0,\tilde{k}-1}, \ldots, u_{0,\tilde{k}'}) \tag{38}$$

of orders $\tilde{n}, \tilde{n}', \tilde{k}, \tilde{k}'$ will exist. With this assumption we shall obtain four conservation laws

$$(T_1 - 1)p_{0,0}^{(m)} = (T_2 - 1)q_{0,0}^{(m)}, \qquad m = 1, 2, 3, 4, \tag{39}$$

with $p_{0,0}^{(m)}$ or $q_{0,0}^{(m)}$ expressed in terms of equation (6).

Theorem 3. *Let equation (6) possess two generalized symmetries (38) and (10). Then equation (6) admits the conservation laws (39):*

$$n > 0, \quad \tilde{n} = n + 1 \quad \Longrightarrow \quad m = 1, \qquad p_{0,0}^{(1)} = \log f_{u_{1,0}}; \tag{40}$$

$$n' < 0, \quad \tilde{n}' = n' - 1 \quad \Longrightarrow \quad m = 2, \qquad p_{0,0}^{(2)} = \log \frac{f_{u_{0,0}}}{f_{u_{0,1}}}; \tag{41}$$

$$k > 0, \quad \tilde{k} = k + 1 \quad \Longrightarrow \quad m = 3, \qquad q_{0,0}^{(3)} = \log f_{u_{0,1}}; \tag{42}$$

$$k' < 0, \quad \tilde{k}' = k' - 1 \quad \Longrightarrow \quad m = 4, \qquad q_{0,0}^{(4)} = \log \frac{f_{u_{0,0}}}{f_{u_{1,0}}}. \tag{43}$$

Proof. Let us consider in detail just the case when $n > 0, \tilde{n} = n + 1$. Due to theorem 2 equation (33) must be satisfied and consequently

$$\left(T_1^{n+1} - 1\right)p_{0,0}^{(1)} = (1 - T_2)T_1 \log \tilde{g}_{u_{n+1,0}}, \tag{44}$$

where $p_{0,0}^{(1)}$ is given by (40). Applying the operator $-T_1$ to equation (33) and adding the result to equation (44), we get the conservation law (39) with $m = 1$, where $q_{0,0}^{(1)}$ is given by

$$q_{0,0}^{(1)} = T_1^2 \log g_{u_{n,0}} - T_1 \log \tilde{g}_{u_{n+1,0}}.$$

The other cases are proved in an analogous way. □

So for equation (6) we have four necessary conditions of integrability: there must exist some functions of a finite range $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}, p_{0,0}^{(4)}$ of the form (18) satisfying the conservation laws (39) with $p_{0,0}^{(1)}, p_{0,0}^{(2)}, q_{0,0}^{(3)}, q_{0,0}^{(4)}$ defined by equations (40)–(43).

The following theorem will precise the structure of the unknown functions $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}$, and $p_{0,0}^{(4)}$.

Theorem 4. *If the functions $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}$ and $p_{0,0}^{(4)}$ satisfy equation (39), with $p_{0,0}^{(1)}, p_{0,0}^{(2)}, q_{0,0}^{(3)}$ and $q_{0,0}^{(4)}$ given by equations (40)–(43), and are written in the form (18) then $q_{0,0}^{(1)}$ and $q_{0,0}^{(2)}$ may depend only on the variables $u_{i,0}$ and $p_{0,0}^{(3)}$ and $p_{0,0}^{(4)}$ on $u_{0,j}$.*

Proof. Let us consider equation (39) with $m = 1$. The functions therein involved have the following form:

$$p_{0,0}^{(1)} = P^{(1)}(u_{1,0}, u_{0,0}, u_{0,1}), \qquad p_{1,0}^{(1)} = P^{(1)}(u_{2,0}, u_{1,0}, u_{1,1}),$$
$$q_{0,0}^{(1)} = Q^{(1)}(u_{N,0}, \ldots, u_{N',0}, u_{0,K}, \ldots, u_{0,K'}),$$
$$q_{0,1}^{(1)} = Q^{(1)}(u_{N,1}, \ldots, u_{N',1}, u_{0,K+1}, \ldots, u_{0,K'+1}).$$

Let us consider the function $q_{0,0}^{(1)}$ and let us study its dependence on the variables $u_{0,j}$ with $j \neq 0$. Using proposition 2, we see that the functions $u_{i,1}$ in $p_{1,0}^{(1)}, q_{0,1}^{(1)}$ may depend only on $u_{0,1}$. If $K > 0$, we differentiate equation (39) with $m = 1$ with respect to $u_{0,K+1}$ and get $\partial_{u_{0,K+1}} q_{0,1}^{(1)} = T_2 \partial_{u_{0,K}} q_{0,0}^{(1)} = 0$. Then, from proposition 1, it follows that $q_{0,0}^{(1)}$ does not depend on $u_{0,K}$. If $K' < 0$, let us differentiate with respect to $u_{0,K'}$ and we get $\partial_{u_{0,K'}} q_{0,0}^{(1)} = 0$. This shows that the function $q_{0,0}^{(1)}$ cannot depend on $u_{0,j}$ with $j \neq 0$.

The proof for the other cases is quite similar. □

As we cannot use the formal variational derivative, we have to work directly with functions $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}, p_{0,0}^{(4)}$ which have the following structure:

$$q_{0,0}^{(m)} = Q^{(m)}\big(u_{N_m,0}, \ldots, u_{N'_m,0}\big), \qquad m = 1, 2;$$
$$p_{0,0}^{(l)} = P^{(l)}\big(u_{0,K_l}, \ldots, u_{0,K'_l}\big), \qquad l = 3, 4.$$

In section 5, we are going to limit ourselves to just five-point symmetries. This will make the problem more definite in the sense that the numbers N_m, N'_m, K_l, K'_l will be specified and small.

5. Integrability conditions for five-point symmetries

From the definition of the Lie symmetry, we can construct a new symmetry by adding the right-hand sides of two symmetries $u_{0,0,t} = g_{0,0}$ and $u_{0,0,\tilde{t}} = \tilde{g}_{0,0}$: $u_{0,0,\hat{t}} = \hat{g}_{0,0} = c_1 g_{0,0} + c_2 \tilde{g}_{0,0}$, where c_1 and c_2 are arbitrary constants. For example, equation (26) of section 3 has two three-point symmetries (27) and (28); therefore, it has a five-point generalized symmetry:

$$u_{0,0,t} = g_{0,0} = G(u_{1,0}, u_{-1,0}, u_{0,0}, u_{0,1}, u_{0,-1}), \qquad g_{u_{1,0}} g_{u_{-1,0}} g_{u_{0,1}} g_{u_{0,-1}} \neq 0. \tag{45}$$

The other known integrable examples of the form (6) have also five points generalized symmetries. We are going to use the existence of a five-point generalized symmetry of the form (45) as an *integrability criterion*. This may be a severe restriction, as there might be integrable equations with symmetries depending on more lattice points.

In the ABS classification, all three-point generalized symmetries turn out to be Miura transformations of the Volterra equation or of the Yamilov discretization of the Krichever–Novikov equation [32]. If we expect to find a new type of integrable discrete equations of the form (6), these should have as generalized symmetries some new type of integrable equations. One example of such an equation is given by the Narita–Itoh–Bogoyavlensky [13, 27, 39] equation

$$u_{0,0,t} = g_{0,0} = u_{0,0}(u_{2,0} + u_{1,0} - u_{-1,0} - u_{-2,0}). \tag{46}$$

We will prove in the appendix that no equation of the form (6) can have equation (46) as a symmetry.

We can then state the following theorem.

Theorem 5. *If equations (6) and (8) possess a generalized symmetry of the form (45), then the functions*

$$\begin{aligned} q_{0,0}^{(m)} &= Q^{(m)}(u_{2,0}, u_{1,0}, u_{0,0}), \qquad m = 1, 2;\\ p_{0,0}^{(m)} &= P^{(m)}(u_{0,2}, u_{0,1}, u_{0,0}), \qquad m = 3, 4, \end{aligned} \tag{47}$$

satisfy conditions (39), (40)–(43).

Proof. From relations (33)–(36), as $n = k = 1$ and $n' = k' = -1$, we are able to construct the functions

$$\begin{aligned} q_{0,0}^{(1)} &= -T_1 \log g_{u_{1,0}}, \qquad & q_{0,0}^{(2)} &= T_1 \log g_{u_{-1,0}},\\ p_{0,0}^{(3)} &= -T_2 \log g_{u_{0,1}}, \qquad & p_{0,0}^{(4)} &= T_2 \log g_{u_{0,-1}}, \end{aligned} \tag{48}$$

satisfying conditions (39), (40)–(43). It follows from equations (22) and (45) that the function $q_{0,0}^{(1)}$ has the structure

$$q_{0,0}^{(1)} = \hat{Q}^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}, u_{1,1}, u_{1,-1}) = Q^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}, u_{0,1}, u_{0,-1}).$$

In analogy to theorem 4 we get that $Q^{(1)}$ cannot depend on $u_{0,1}, u_{0,-1}$. The proof for the other functions contained in equations (48) is obtained in the same way. □

So, for a given equation (6), we check the integrability conditions (39), (40)–(43) with the unknown functions $q_{0,0}^{(m)}$ and $p_{0,0}^{(m)}$ given in the form (47). If the integrability conditions are satisfied, we can construct the most general unknown functions $q_{0,0}^{(m)}$ and $p_{0,0}^{(m)}$ of the form (47) and then, from equations (48), build the partial derivatives of $g_{0,0}$. The partial derivatives of $g_{0,0}$ must be consistent. The consistency of equations (48) implies that the *additional integrability conditions*

$$g_{u_{1,0},u_{-1,0}} = g_{u_{-1,0},u_{1,0}}, \qquad g_{u_{0,1},u_{0,-1}} = g_{u_{0,-1},u_{0,1}} \tag{49}$$

must be satisfied. If equations (49) are satisfied, we obtain the right-hand side of the symmetry (45) up to an arbitrary unknown function of $u_{0,0}$ of the form $\phi(u_{0,0})$. The function ϕ is derived by using compatibility condition (12), the *final integrability condition*.

The function $g_{0,0}$, so obtained, will thus be of the form

$$g_{0,0} = \Phi(u_{1,0}, u_{0,0}, u_{-1,0}) + \Psi(u_{0,1}, u_{0,0}, u_{0,-1}), \tag{50}$$

i.e. the right-hand side of any five-point symmetry (45) must have the form (50). The same result has been obtained by Rasin and Hydon in [47].

All the known integrable autonomous equations (6) have symmetries of the following two types:

$$\Psi = 0 \qquad \text{and} \qquad \Phi_{u_{1,0}}\Phi_{u_{-1,0}} \neq 0; \tag{51}$$

$$\Phi = 0 \qquad \text{and} \qquad \Psi_{u_{0,1}}\Psi_{u_{0,-1}} \neq 0. \tag{52}$$

Thus any symmetry of the form (45) and (50) is the linear combination of a symmetry (51) and (52). However, we cannot prove this property theoretically.

Obviously, the scheme described in this section and in the previous sections can also be applied to the simpler symmetries (51) and (52). For example, in the case of a symmetry given by equations (50) and (51), the integrability conditions (39)–(41) must be satisfied. The first two equations of equation (48) allow us to construct the partial derivatives of $g_{0,0} = \Phi$. Then we check the first of conditions (49). If it is satisfied, we can find Φ up to an arbitrary function $\phi(u_{0,0})$, which can be specified by using the compatibility condition (12).

In the case of the example considered in equation (26) in section 3, it is easy to check that conditions (39), (40)–(43) are satisfied. Moreover, using the generalized symmetries (27), (28) and equations (48), we easily construct four conservation laws which are linear combinations with shift-dependent parameters of the conservation laws (29) and (30).

It should be remarked that the integrability conditions analogous to equations (39) and (40)–(43) have been derived for the hyperbolic systems of the form (4) by Zhiber and Shabat in [61].

6. A simple classification problem

Here we apply the formulas introduced before to study the class of equations:

$$u_{1,1} = f_{0,0} = u_{1,0} + u_{0,1} + \varphi(u_{0,0}). \tag{53}$$

This will be just an example of how to use the integrability conditions. More interesting classification problems, as well as the possible discovery of new integrable discrete equations, are left for the future. The class of equations (53) depends on an unknown function φ, and we require that equation (53) possesses a generalized symmetry of the form (45). To do so it must satisfy the integrability conditions (39), (40)–(43) and (47). If $\varphi'' = 0$, equation (53) is linear, and all the integrability conditions are satisfied trivially. So we require that $\varphi'' \neq 0$.

The proof that equations (39), (40)–(43) and (47) are conservation laws is carried out by differentiating them in such a way to reduce them to simple differential equations, a scheme introduced in 1823 by Abel [1] (see [4] for a review) for solving functional equations. The applications of this scheme for the difference equation can be found in [25, 34, 48]. In [48] the scheme was used for finding conservation laws for known equations, i.e. when the dependence of the functions $p_{0,0}$ and $q_{0,0}$ on the symmetries and on equation (6) was unknown while the difference equation (6) was given. In [49] the existence of a simple conservation law is used as an integrability condition.

Here we consider the case when either $p_{0,0}$ or $q_{0,0}$ is expressed in terms of the unknown right-hand side of equation (6). The conservation laws are allowed to depend on the arbitrary functions of the variables $u_{1,0}, u_{0,0}, u_{0,1}$. Moreover, as will be shown at the end of this section, the existence of simple conservation laws is not sufficient to prove integrability. One can have the nonlinear equations of this class (53) with two local conservation laws but with no generalized symmetry.

Let us study the class of difference equations (53). For later use, we can rewrite equation (53) in three equivalent forms, applying to it the operators T_1^{-1}, T_2^{-1}:

$$\begin{aligned} u_{-1,1} &= u_{0,1} - u_{0,0} - \varphi(u_{-1,0}), \\ u_{1,-1} &= u_{1,0} - u_{0,0} - \varphi(u_{0,-1}), \\ u_{-1,-1} &= \varphi^{-1}(u_{0,0} - u_{-1,0} - u_{0,-1}). \end{aligned} \tag{54}$$

Let us consider condition (39) with $m = 2$. Applying the shift operators T_1^{-1}, T_2^{-1}, we rewrite it in two equivalent forms

$$p^{(2)}_{0,0} - p^{(2)}_{-1,0} = q^{(2)}_{-1,1} - q^{(2)}_{-1,0}, \tag{55}$$

$$p^{(2)}_{0,-1} - p^{(2)}_{-1,-1} = q^{(2)}_{-1,0} - q^{(2)}_{-1,-1}, \tag{56}$$

where $p^{(2)}_{0,0} = \log\varphi'(u_{0,0})$ and $q^{(2)}_{0,0}$ is given by equation (47). Taking into account equations (53) and (54), equations (55) and (56) can be expressed in terms of the independent variables (9).

Equations (55) and (56) are two functional equations for $q^{(2)}_{0,0}$. By applying the following operators

$$\hat{\mathcal{A}} = \partial_{u_{0,0}} + \partial_{u_{1,0}} + \partial_{u_{-1,0}}, \qquad \hat{\mathcal{B}} = \partial_{u_{0,0}} - \varphi'(u_{0,0})\partial_{u_{1,0}} - \frac{1}{\varphi'(u_{-1,0})}\partial_{u_{-1,0}},$$

we reduce them to the partial differential equations. Using equations (54), we can show that $\hat{\mathcal{A}}$ annihilates any function $\Phi(u_{1,-1}, u_{0,-1}, u_{-1,-1})$. So, applying $\hat{\mathcal{A}}$ to equation (56), we get

$$\hat{\mathcal{A}} q^{(2)}_{-1,0} = 0. \tag{57}$$

The operator $\hat{\mathcal{B}}$ annihilates $q^{(2)}_{-1,1}$. Thus, applying the operator $\hat{\mathcal{B}}$ to equation (55), we get

$$\hat{\mathcal{B}} q^{(2)}_{-1,0} = -\hat{\mathcal{B}}\big(p^{(2)}_{0,0} - p^{(2)}_{-1,0}\big).$$

If we introduce the difference operator $\hat{\mathcal{C}} = \hat{\mathcal{A}} - \hat{\mathcal{B}}$, we get

$$\hat{\mathcal{C}} q^{(2)}_{-1,0} = \hat{\mathcal{B}}\big(p^{(2)}_{0,0} - p^{(2)}_{-1,0}\big). \tag{58}$$

From equations (57) and (58) we also get

$$[\hat{\mathcal{A}}, \hat{\mathcal{C}}] q^{(2)}_{-1,0} = \hat{\mathcal{A}}\hat{\mathcal{B}}\big(p^{(2)}_{0,0} - p^{(2)}_{-1,0}\big), \tag{59}$$

where $[\hat{\mathcal{A}}, \hat{\mathcal{C}}]$ is the standard commutator of two operators. So equations (57)–(59) can be rewritten as a partial differential system for the function $q = q^{(2)}_{-1,0}$, where, as before, by the indexes we denote the partial derivatives and by apices derivatives with respect to the argument

$$\begin{aligned} &q_{u_{0,0}} + q_{u_{1,0}} + q_{u_{-1,0}} = 0,\\ &a(u_{0,0}) q_{u_{1,0}} + b(u_{-1,0}) q_{u_{-1,0}} = c(u_{0,0}) - b'(u_{-1,0}),\\ &a'(u_{0,0}) q_{u_{1,0}} + b'(u_{-1,0}) q_{u_{-1,0}} = c'(u_{0,0}) - b''(u_{-1,0}). \end{aligned} \tag{60}$$

The functions $a(z)$, $b(z)$ and $c(z)$ are given by

$$a(z) = \varphi'(z) + 1, \qquad b(z) = \frac{1}{\varphi'(z)} + 1, \qquad c(z) = \frac{\varphi''(z)}{\varphi'(z)},$$

where $a'(z)b'(z)c(z) \neq 0$, as $\varphi''(z) \neq 0$.

The solvability of the system (60) depends on the following determinant:

$$\Delta = \begin{vmatrix} a(u_{0,0}) & b(u_{-1,0}) \\ a'(u_{0,0}) & b'(u_{-1,0}) \end{vmatrix}.$$

We must have $\Delta \neq 0$. If we have $\Delta = 0$, as $u_{0,0}$ and $u_{-1,0}$ are independent variables, we obtain the relations $\frac{a'(u_{0,0})}{a(u_{0,0})} = \frac{b'(u_{-1,0})}{b(u_{-1,0})} = \nu$, where ν is a constant. These relations are in contradiction with the condition that $\varphi'' \neq 0$.

If we differentiate the system (60) with respect to $u_{1,0}$, we easily deduce that $q_{u_{1,0}} = \alpha$, where α is a constant. Then from equations (60) we obtain two different expressions for $q_{u_{-1,0}}$:

$$q_{u_{-1,0}} = \frac{d(u_{0,0}) - b'(u_{-1,0})}{b(u_{-1,0})} = \frac{d'(u_{0,0}) - b''(u_{-1,0})}{b'(u_{-1,0})}, \qquad d(z) = c(z) - \alpha a(z). \tag{61}$$

If $d' \neq 0$, differentiating equation (61) with respect to $u_{0,0}$, we get $\frac{d''(u_{0,0})}{d'(u_{0,0})} = \frac{b'(u_{-1,0})}{b(u_{-1,0})} = \sigma$, where σ is a constant. This result is again in contradiction with the condition $\varphi'' \neq 0$. So, $d = \beta$, a constant, and we get the following ODE for φ:

$$\varphi''/\varphi' = \alpha\varphi' + \alpha + \beta. \tag{62}$$

If φ satisfies equation (62), the condition (61) is satisfied, and $q_{u_{-1,0}} = \alpha + \beta$.

The system (60) provides us with another partial derivative of q:

$$q_{u_{0,0}} = -2\alpha - \beta,$$

from which we deduce that

$$q = q^{(2)}_{-1,0} = \alpha u_{1,0} - (2\alpha + \beta)u_{0,0} + (\alpha + \beta)u_{-1,0} + \delta,$$

where δ is an arbitrary constant. The integration of equation (62) gives

$$\log \varphi'(z) = \alpha\varphi(z) + (\alpha + \beta)z + \gamma,$$

where γ is a further constant. If we introduce these last two equations into equation (55), we get $\beta u_{0,0} + \beta\varphi(u_{-1,0}) = 0$, which implies $\beta = 0$.

Thus, we have proved that equation (53) satisfies the condition (39) with $m = 2$ if and only if

$$\log \varphi'(z) = \alpha(\varphi(z) + z) + \gamma, \tag{63}$$

with $\alpha \neq 0$, as $\varphi'' \neq 0$. Equation (63) can easily be solved, but its solution is not particularly relevant to the economy of this paper. If equation (63) is satisfied,

$$p^{(2)}_{0,0} = \log \varphi'(u_{0,0}), \qquad q^{(2)}_{0,0} = \alpha(u_{2,0} - 2u_{1,0} + u_{0,0}) + \delta \tag{64}$$

and these functions define a nontrivial conservation law.

If equation (53), with φ given by equation (63), has a generalized symmetry of the form (45), the other conditions (39), (40)–(43), (47) must be satisfied. From equation (40), we get that the condition (39) with $m = 1$ becomes $(T_2 - 1)q^{(1)}_{0,0} = 0$. This equation has a trivial solution, $q^{(1)}_{0,0}$ a constant. We now look for a nontrivial solution. From equations (47) it follows that the functions $q^{(1)}_{0,0}$ and $q^{(2)}_{0,0}$ depend on the same set of variables. Hence $\tilde{q} = q^{(1)}_{-1,0}$ also satisfies equations (60), but with zeros on the right-hand side. As $q_{u_{1,0}}$ is a constant, it follows that also $q^{(1)}_{0,0}$ must be a constant, i.e. the constant solution is the most general one. From equations (48), we get the partial derivatives of the right-hand side of the symmetry (45), $g_{u_{1,0}}$ and $g_{u_{-1,0}}$. It is easy to verify that the first of the conditions (49) is not satisfied. Consequently, equation (53), with φ given by equation (63), has no generalized symmetry of the form (45).

In section 5, we have considered the simpler symmetries (51) and (52). Using the previous reasoning, we can prove that there is no symmetry defined by equations (50) and (51). Equation (53) is symmetric under the involution $u_{i,j} \to u_{j,i}$. Also the conditions (39) with $m = 3, 4$ are symmetric with respect to the conditions (39) with $m = 1, 2$. So these further conditions will provide a conservation law symmetric to the one defined by equations (64) and prove that there is no symmetry given by equations (50) and (52).

Let us collect the results obtained so far in the following theorem, where the conservation laws will be written in a simplified form, omitting inessential constants.

Theorem 6. *Equation (53) satisfies the integrability conditions (39), (40)–(43), (47) iff φ is a solution of equation (63). Equation (53), when φ is given by equation (63), has two nontrivial conservation laws:*

$$\begin{aligned} (T_1 - 1)(\varphi(u_{0,0}) + u_{0,0}) &= (T_2 - 1)(u_{2,0} - 2u_{1,0} + u_{0,0}), \\ (T_2 - 1)(\varphi(u_{0,0}) + u_{0,0}) &= (T_1 - 1)(u_{0,2} - 2u_{0,1} + u_{0,0}). \end{aligned} \tag{65}$$

However, in this case, equation (53) does not have a generalized symmetry of the form (45) or of the form given by equations (50), (51) or (50), (52).

Let us note that equation (53) possesses the conservation laws (65) for any φ, not only when φ satisfies equation (63). However, the integrability conditions are satisfied only if φ satisfies equation (63), but no generalized symmetry of the form mentioned in theorem 6 exists.

7. Conclusions

In this paper, we have considered the classification problem for difference equations by asking for the existence of a generalized symmetry. In this way, we have obtained the lowest order integrability conditions which turn out to be written as conservation laws. The verification of the existence of finite-order conservation laws is in general a very complicated problem due to the high number of unknown involved. So we limited ourselves to the case when we have just a five-point symmetry. In this case, we easily can find some further integrability conditions which make our problem solvable. At the end, we present an example of classification when we have just an arbitrary function of one variable.

This research is far from complete. At the moment, we are working on

(i) obtaining further integrability conditions by adding extra structures;
(ii) applying the result contained in this work for testing the integrability of some discrete equations of the class (1) as, for example, Q_V [54];
(iii) classifying equations (1) in the case of an arbitrary function of two variables.

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Appendix

Theorem 7. *No equation of the forms (6) and (8) can have a generalized symmetry of the form of equation (46).*

Proof. We use conditions (33) and (34) with $n = 2$ and $n' = -2$. Applying the operators T_1^{-1} and $-T_1$, we rewrite them in the form

$$p^{(1)}_{1,0} - p^{(1)}_{-1,0} = \log \frac{u_{0,0}}{u_{0,1}}, \tag{A.1}$$

$$p^{(2)}_{1,0} - p^{(2)}_{-1,0} = \log \frac{u_{1,1}}{u_{1,0}}, \tag{A.2}$$

where $p^{(1)}_{0,0}$ and $p^{(2)}_{0,0}$ are given by equations (40) and (41). Studying conditions (A.1) and (A.2), we will use in addition to equation (6) its equivalent form

$$u_{-1,1} = \hat{f}_{0,0} = \hat{F}(u_{-1,0}, u_{0,0}, u_{0,1}).$$

The functions $p^{(m)}_{0,0}$ have the structure $p^{(m)}_{0,0} = P^{(m)}(u_{1,0}, u_{0,0}, u_{0,1})$. Therefore, $p^{(m)}_{-1,0} = P^{(m)}(u_{0,0}, u_{-1,0}, \hat{f}_{0,0})$ and the right-hand sides of equations (A.1) and (A.2) do not depend on $u_{2,0}$. The functions $p^{(m)}_{1,0} = P^{(m)}(u_{2,0}, u_{1,0}, f_{0,0})$ depend on $u_{2,0}$, and from equations (A.1) and (A.2) we get $\partial_{u_{2,0}} p^{(m)}_{1,0} = T_1 \partial_{u_{1,0}} p^{(m)}_{0,0} = 0$. Moreover, according to proposition 1,

$$\partial_{u_{1,0}} p^{(m)}_{0,0} = 0, \qquad m = 1, 2. \tag{A.3}$$

□

From equation (A.3) with $m = 1$, we get $f_{u_{1,0}u_{1,0}} = 0$, i.e. $f_{0,0}$ can be expressed as

$$f_{0,0} = a_{0,0}u_{1,0} + b_{0,0} = A(u_{0,0}, u_{0,1})u_{1,0} + B(u_{0,0}, u_{0,1}), \tag{A.4}$$

where $a_{0,0} \neq 0$ due to condition (8). Now $p^{(1)}_{0,0} = \log a_{0,0}$ and equation (A.1) is rewritten as

$$\frac{a_{1,0}}{a_{-1,0}} = \frac{u_{0,0}}{u_{0,1}}. \tag{A.5}$$

Here only the function $a_{1,0}$ depends on $u_{1,0}$, and we get

$$\frac{\mathrm{d}a_{1,0}}{\mathrm{d}u_{1,0}} = \partial_{u_{1,0}}a_{1,0} + a_{0,0}\partial_{u_{1,1}}a_{1,0} = 0.$$

Applying to it the shift operator T_1^{-1}, we get the more convenient form

$$\partial_{u_{0,0}}a_{0,0} + a_{-1,0}\partial_{u_{0,1}}a_{0,0} = 0. \tag{A.6}$$

As $a_{-1,0} \neq 0$, only two cases are possible. The first one is when $\partial_{u_{0,0}}a_{0,0} = \partial_{u_{0,1}}a_{0,0} = 0$, i.e. $a_{0,0}$ is a constant. This is in contradiction with equation (A.5). So, $\partial_{u_{0,0}}a_{0,0} \neq 0$ and $\partial_{u_{0,1}}a_{0,0} \neq 0$.

From equation (A.3) with $m = 2$ we get

$$\frac{f_{u_{0,0}u_{1,0}}}{f_{u_{0,0}}} - \frac{f_{u_{0,1}u_{1,0}}}{f_{u_{0,1}}} = 0.$$

Using this equation together with equations (A.4) and (A.6), we get

$$p^{(2)}_{0,0} = \log\frac{f_{u_{0,0}u_{1,0}}}{f_{u_{0,1}u_{1,0}}} = \log\frac{\partial_{u_{0,0}}a_{0,0}}{\partial_{u_{0,1}}a_{0,0}} = \log(-a_{-1,0}).$$

Applying T_1 we can rewrite equation (A.2) as

$$\frac{a_{1,0}}{a_{-1,0}} = \frac{u_{2,1}}{u_{2,0}}. \tag{A.7}$$

Comparing equations (A.5) and (A.7) and using equation (A.4), we get $u_{2,1} = \frac{u_{0,0}}{u_{0,1}}u_{2,0} = a_{1,0}u_{2,0} + b_{1,0}$. As $a_{1,0}$ and $b_{1,0}$ do not depend on $u_{2,0}$, we obtain from here $a_{1,0} = \frac{u_{0,0}}{u_{0,1}}$. Then from equation (A.5), we obtain $a_{-1,0} = 1$. These two last results are in contradiction, thus proving the theorem.

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